Mathematical Logic

Part Three

The Aristotelian Forms

"All *A*s are *B*s"

$$\forall x. (A(x) \to B(x))$$

"Some *A*s are *B*s"

$$\exists x. (A(x) \land B(x))$$

"No *A*s are *B*s"

$$\forall x. (A(x) \to \neg B(x))$$

"Some *A*s aren't *B*s"

$$\exists x. (A(x) \land \neg B(x))$$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

The Art of Translation

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means “everybody loves someone else.”

How many of the following first-order logic statements are correct translations of "**everyone loves someone else**?"

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \land \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \to Loves(p, q)))$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(p, q)))$$

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, or **4**.

Everybody loves someone else

Every person loves some other person

Every person p loves some other person

Every person p loves some other person

"All As are Bs"

$$\forall x.\ (A(x) \rightarrow B(x))$$

$\forall p.\ (Person(p) \rightarrow$

p loves some other person

$)$

"All As are Bs"

$\boldsymbol{\forall x.\ (A(x) \rightarrow B(x))}$

$\forall p.\ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p.\ (Person(p) \to$

there is some other person that p loves

$)$

$\forall p. \ (Person(p) \rightarrow$

there is a person other than p that p loves

$)$

$\forall p.\ (Person(p) \rightarrow$

there is a person q, other than p, where p loves q

$)$

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"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x) \land B(x))}$

$\forall p. \ (Person(p) \rightarrow$

$\exists q. \ (Person(q) \ \wedge$*, other than p, where*

p loves q

$)$

$)$

"Some As are Bs"

$\boldsymbol{\exists x. \ (A(x) \ \wedge \ B(x))}$

$\forall p.\ (Person(p) \rightarrow$

$\exists q.\ (Person(q)\ \land$*, other than p, where*

p loves q

$)$

$)$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q) \land p \neq q \land$

p loves q

$\quad)$

$)$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

Using the predicates

- *Person*(p), which states that p is a person, and
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write a sentence in first-order logic that means “there is a person that everyone else loves.”

There is a person that everyone else loves

There is a person p where everyone else loves p

There is a person p where everyone else loves p

"Some *A*s are *B*s"

$$\exists x.\ (A(x) \land B(x))$$

$\exists p.\ (Person(p)\ \wedge$

everyone else loves p

$)$

"Some *A*s are *B*s"

$\boldsymbol{\exists x.\ (A(x)\ \wedge\ B(x))}$

$\exists p.\ (Person(p)\ \wedge$

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$)$

$\exists p.\ (Person(p)\ \land$

every other person q loves p

$)$

$\exists p.\ (Person(p)\ \wedge$

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"All As are Bs"

$\boldsymbol{\forall x.\ (A(x) \to B(x))}$

$\exists p.\ (Person(p) \land$

$\quad \forall q.\ (Person(q) \land p \neq q \rightarrow$

$\quad\quad$ *q loves p*

$\quad)$

$)$

"All *A*s are *B*s"

$\boldsymbol{\forall x.\ (A(x) \rightarrow B(x))}$

$$
\begin{array}{l}
\exists p.\ (Person(p)\ \land \\
\quad \forall q.\ (Person(q)\ \land\ p \neq q \to \\
\qquad \textit{q loves p} \\
\quad) \\
)
\end{array}
$$

$$
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Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
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there is some person

who isn't them

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$$\forall p.\ (Person(p) \rightarrow \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

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who isn't them

that they love.

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who everyone

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There is some person

who everyone

who isn't them

loves.

For Comparison

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

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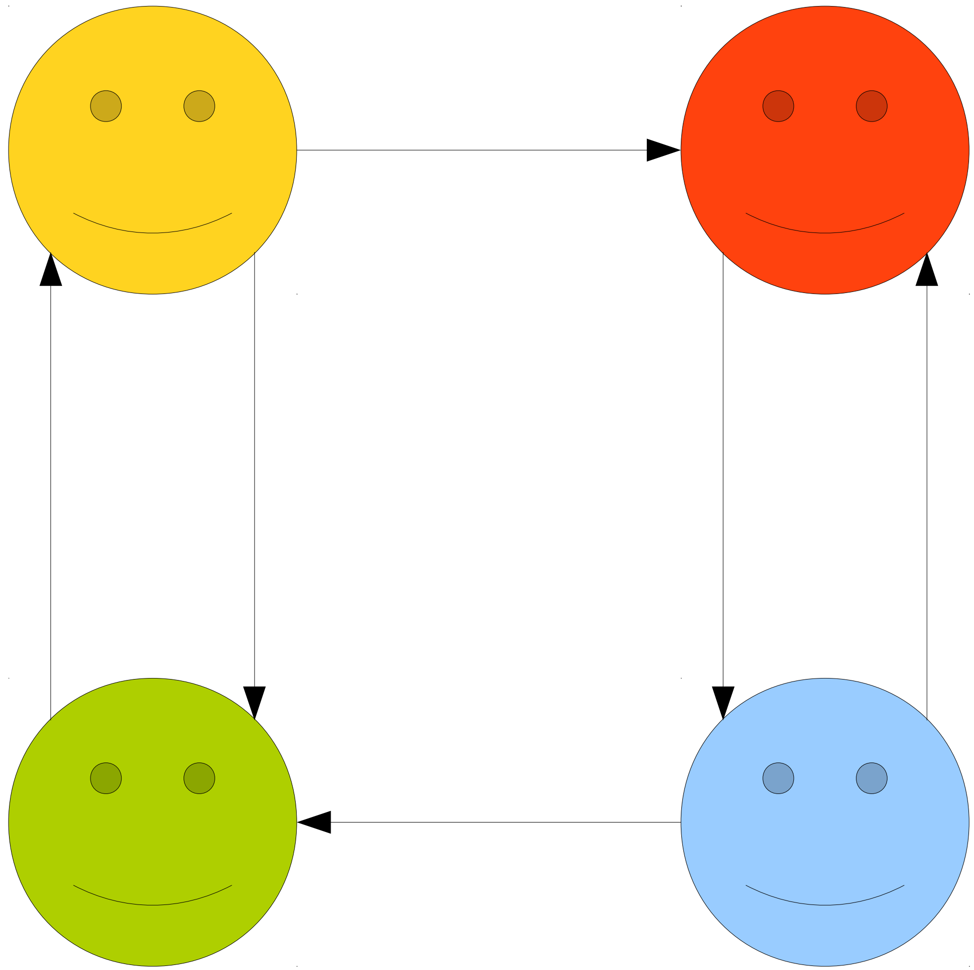
There is some person

who everyone

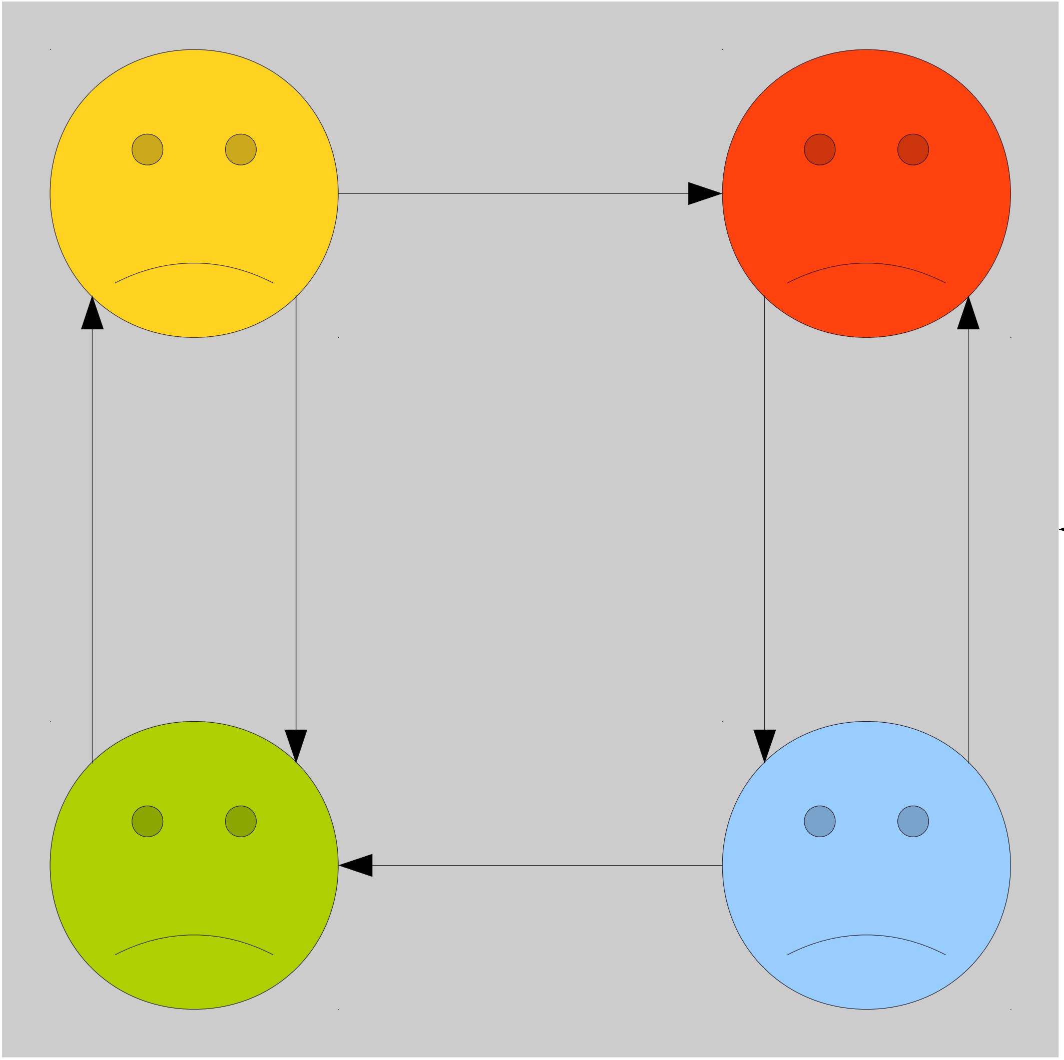
who isn't them

loves.

Everyone Loves Someone Else

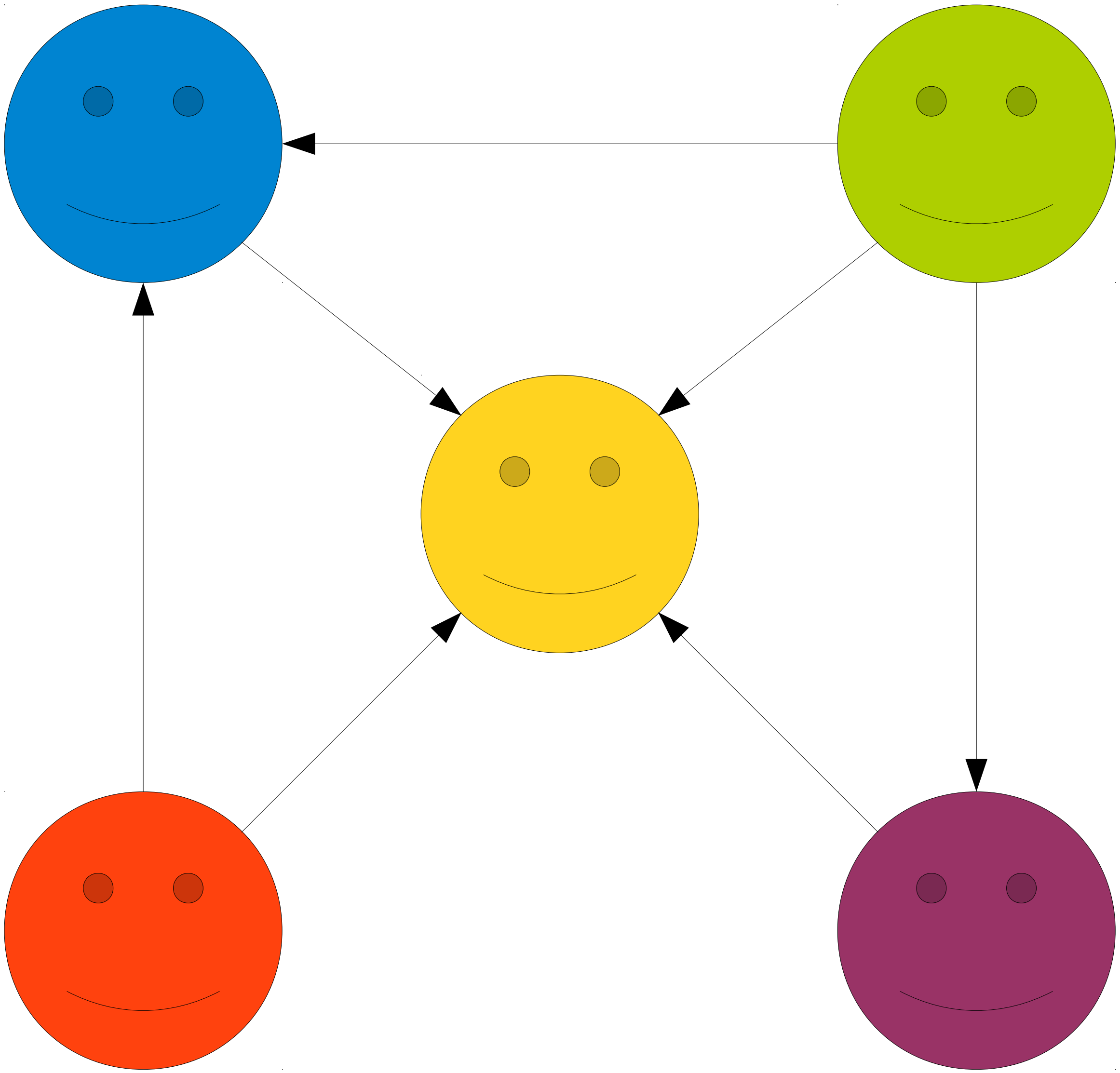


Everyone Loves Someone Else

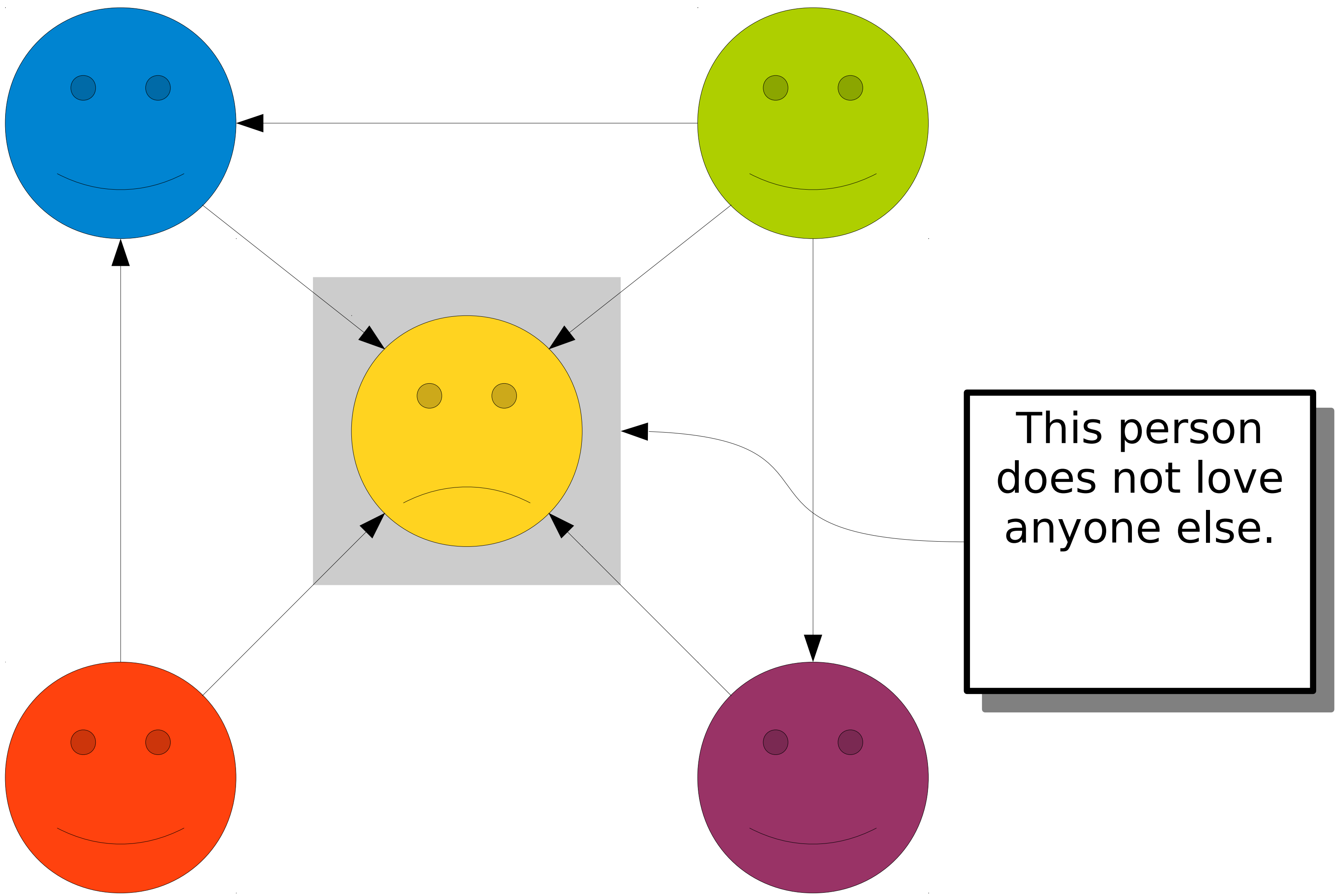


No one here is universally loved.

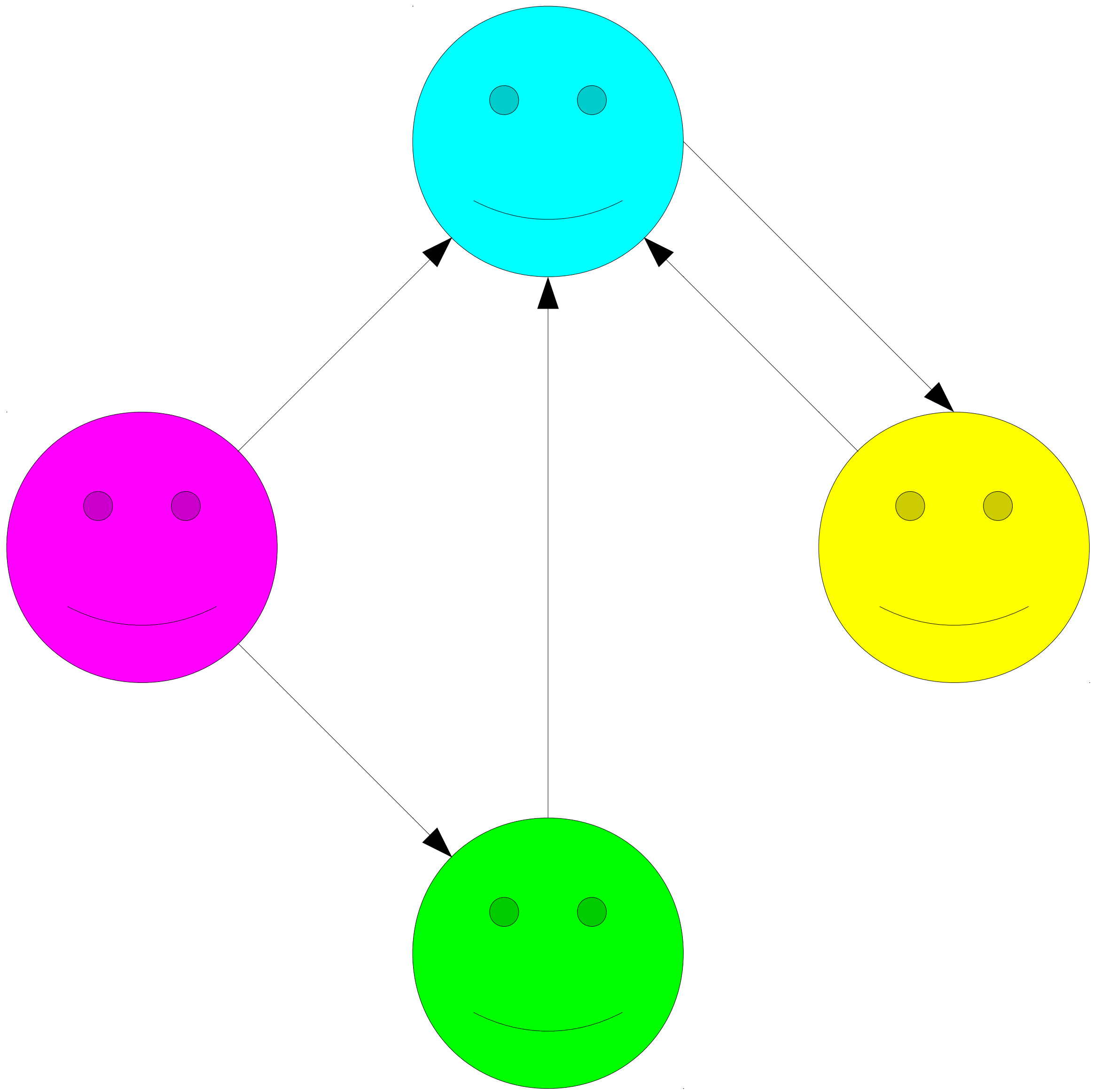
There is Someone Everyone Else Loves



There is Someone Everyone Else Loves



Everyone Loves Someone Else ***and*** There is Someone Everyone Else Loves



$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\land$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

There is some person

who everyone

who isn't them

loves.

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

  means "for any choice of x, there's some choice of y where $P(x, y)$ is true."
- The choice of y can be different every time and can depend on x.

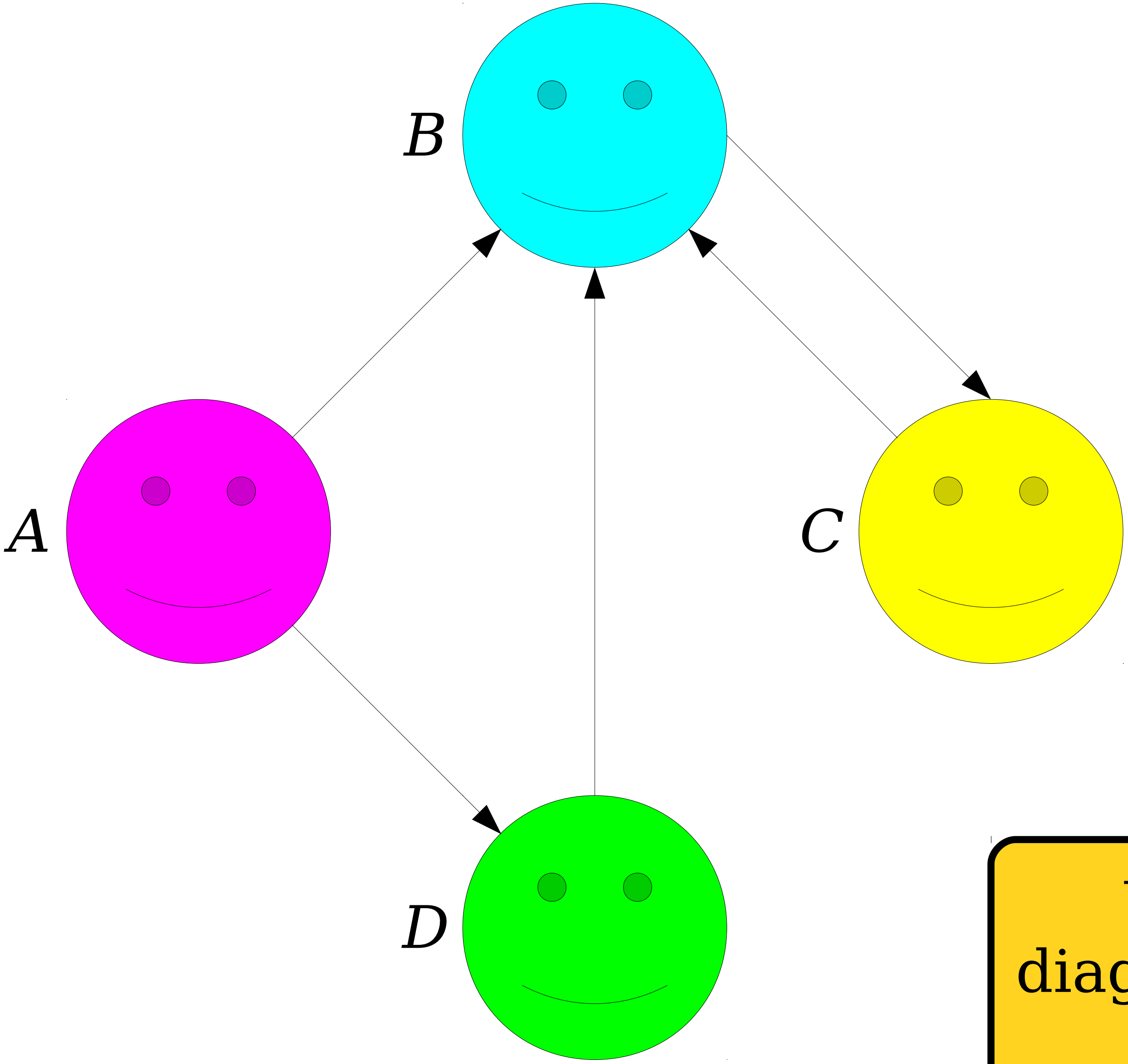
Quantifier Ordering

- The statement

$$\exists x.\ \forall y.\ P(x, y)$$

  means “there is some x where for any choice of y, we get that $P(x, y)$ is true.”
- Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!



Which person in this diagram do you most aspire to be?

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, or **D**.

Back to CS103!

Set Translations

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “the empty set exists.”

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write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

How many of the following first-order logic statements are correct translations of "the empty set exists"?

$$\exists S.\ (Set(S)\ \land \\ \quad \neg\exists x.\ x \in S \\ )$$

$$\exists S.\ (Set(S)\ \land \\ \quad \exists x.\ x \notin S \\ )$$

$$\exists S.\ (Set(S)\ \land \\ \quad \neg\forall x.\ x \in S \\ )$$

$$\exists S.\ (Set(S)\ \land \\ \quad \forall x.\ x \notin S \\ )$$

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The empty set exists.

There is some set S that is empty.

$\exists S.\ (Set(S)\ \wedge$

S is empty.

$)$

$\exists S.\ (Set(S)\ \wedge$

there are no elements in S

$)$

$\exists S.\ (Set(S)\ \wedge$

$\neg$*there is an element in S*

$)$

$\exists S.\ (Set(S)\ \wedge$

$\neg$*there is an element x in S*

$)$

$$
\begin{aligned}
&\exists S.\ (Set(S)\ \wedge \\
&\quad \neg\exists x.\ x \in S \\
&)
\end{aligned}
$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land$

there are no elements in S

$)$

$$\exists S.\ (Set(S) \wedge \neg \exists x.\ x \in S)$$

$\exists S.\ (Set(S) \wedge$

every object does not belong to S

$)$

$$\exists S.\ (Set(S) \wedge \neg \exists x.\ x \in S)$$

$\exists S.\ (Set(S) \wedge$

every object x does not belong to S

$)$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land$$
$$\quad \forall x.\ x \notin S$$
$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

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Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

Mechanics: Negating Statements

Which of the following is the negation of the statement
$\forall x.\ \exists y.\ Loves(x, y)$?

A. $\forall x.\ \forall y.\ \neg Loves(x, y)$
B. $\forall x.\ \exists y.\ \neg Loves(x, y)$
C. $\exists x.\ \forall y.\ \neg Loves(x, y)$
D. $\exists x.\ \exists y.\ \neg Loves(x, y)$
E. None of these.
F. Two or more of these.

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An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For any choice of x, $P(x)$

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$\exists x.\ P(x)$	For some choice of x, $P(x)$	**For any choice of x, $\neg P(x)$**
$\forall x.\ \neg P(x)$	**For any choice of x, $\neg P(x)$**	For some choice of x, $P(x)$
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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
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$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
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$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
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Negating First-Order Statements

- Use the equivalences

$$\neg\forall x.\ A \quad \equiv \quad \exists x.\ \neg A$$

$$\neg\exists x.\ A \quad \equiv \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg\forall x.\ \exists y.\ Loves(x, y)$$
$$\exists x.\ \neg\exists y.\ Loves(x, y)$$
$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\boldsymbol{\neg(p \land q) \quad \equiv \quad p \to \neg q}$$

$$\boldsymbol{\neg(p \to q) \quad \equiv \quad p \land \neg q}$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says “there is a cute puppy”?

$$\boldsymbol{\exists x.\ (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

- This says “no puppy is cute.”
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*"There is a set with no elements."*)

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$

$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

(*"Every set contains at least one element."*)

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*“There is a set that doesn't contain anything”*)

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

(*“Everything is a set that contains something”*)

Remember: ∀ usually goes with →, not ∧

Restricted Quantifiers

Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means "for any element x of set S, $P(x)$ holds." (It's vacuously true if S is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

  means "there is an element x of set S where $P(x)$ holds." (It's false if S is empty.)

Quantifying Over Sets

- The syntax

$$\forall x \in S.\ \varphi$$

$$\exists x \in S.\ \varphi$$

  is allowed for quantifying over sets.
- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x$ with $P(x)$. $Q(x)$ ⚠

⚠ $\forall y$ such that $P(y) \land Q(y)$. $R(y)$. ⚠

⚠ $\exists P(x)$. $Q(x)$ ⚠

Expressing Uniqueness

Using the predicate

- $Level(l)$, which states that l is a level,

write a sentence in first-order logic that means “there is only one level.”

A fun diversion:

http://www.onemorelevel.com/game/there_is_only_one_level

There is only one level.

Something is a level, and nothing else is.

Some thing l is a level, and nothing else is.

Some thing l is a level, and nothing besides l is a level

$\exists l.\ (Level(l)\ \wedge$

nothing besides l is a level.

$)$

$\exists l.\ (Level(l)\ \wedge$

anything that isn't l isn't a level

$)$

$\exists l.\ (Level(l)\ \wedge$

any thing x that isn't l isn't a level

$)$

$\exists l.\ (Level(l)\ \land$

$\quad \forall x.\ (x \neq l \to$ *x isn't a level*$)$

$)$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (x \neq l \to \neg Level(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (x \neq l \rightarrow \neg Level(x)) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (Level(x) \rightarrow x = l) \\
&)
\end{aligned}
$$

Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special “uniqueness quantifier” used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “two sets are equal if and only if they contain the same elements.”

Two sets are equal if and only if they have the same elements.

Any two sets are equal if and only if they have the same elements.

Any two sets S and T are equal if and only if they have the same elements.

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

S and T are equal if and only if they have the same elements.

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T$ *if and only if they have the same elements.*$)$

$\quad)$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *they have the same elements.*)

$\quad)$

$)$

$$
\begin{array}{l}
\forall S.\ (Set(S) \to \\
\quad \forall T.\ (Set(T) \to \\
\qquad (S = T \leftrightarrow \textit{S and T have the same elements.}) \\
\\
\quad) \\
)
\end{array}
$$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *every element of S is an element of T and vice-versa*)

$\quad)$

$)$

$$\forall S.\ (Set(S) \to$$
$$\quad \forall T.\ (Set(T) \to$$
$$\quad\quad (S = T \leftrightarrow \textit{x is an element of S if and only if x is an element of T})$$
$$\quad)$$
$$)$$

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\forall S.\ (Set(S) \to \\
\quad \forall T.\ (Set(T) \to \\
\quad\quad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\forall S.\ (Set(S) \to \\
\quad \forall T.\ (Set(T) \to \\
\quad\quad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
\quad) \\
)
\end{array}
$$

You sometimes see the universal quantifier pair with the ↔ connective. This is especially common when talking about sets because two sets are equal when they have precisely the same elements.

$$
\begin{array}{l}
\forall S.\ (Set(S) \rightarrow \\
\quad \forall T.\ (Set(T) \rightarrow \\
\quad\quad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
\quad) \\
)
\end{array}
$$

Next Time

- ***Binary Relations***
  - How do we model connections between objects?
- ***Equivalence Relations***
  - How do we model the idea that objects can be grouped into clusters?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?